SYDNEY GRAMMAR SCHOOL

2012 Trial Examination

FORM VI
MATHEMATICS EXTENSION 1

Thursday 9th August 2012

General Instructions
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 70 Marks
- All questions may be attempted.

Section I – 10 Marks
- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 60 Marks
- Questions 11–14 are of equal value.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Collection
- Write your candidate number clearly on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist
- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 128 boys

Examiner
SO/MLS

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

What is the remainder when $P(x) = x^2 + 5x + 7$ is divided by $(x + 3)$? **1**

(A) $-\dfrac{109}{9}$

(B) $\dfrac{19}{9}$

(C) 31

(D) 1

QUESTION TWO

The point R divides the interval joining $P(a, 2b)$ and $Q(3a, -b)$ externally in the ratio $2 : 3$. What are the coordinates of R? **1**

(A) $(-3a, 8b)$

(B) $\left(\dfrac{11a}{5}, \dfrac{4b}{5}\right)$

(C) $(7a, -7b)$

(D) $\left(\dfrac{9a}{5}, \dfrac{8b}{5}\right)$

QUESTION THREE

The term independent of x in the expansion of $\left(x + \dfrac{2}{x}\right)^6$ is: **1**

(A) 160

(B) 80

(C) 40

(D) 20

Exam continues next page ...

QUESTION FOUR

A simplified expression for $\binom{n+1}{n-1}$ is: **1**

(A) $\frac{1}{2}(n^2 - n)$

(B) $\frac{1}{2}(n^2 + n)$

(C) $n^2 - n$

(D) $n^2 + n$

QUESTION FIVE

It is given that x_1 is a good approximate solution of $\cos x = x$. Using one step of Newton's method, a better approximation is: **1**

(A) $x_2 = x_1 - \dfrac{\cos x_1 + x_1}{\sin x_1 + 1}$

(B) $x_2 = x_1 - \dfrac{\cos x_1 - x_1}{\sin x_1 - 1}$

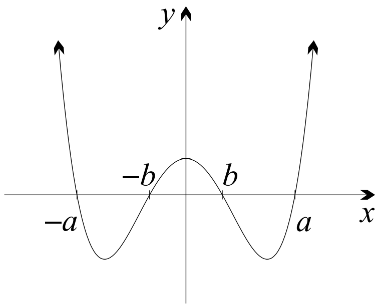
(C) $x_2 = x_1 + \dfrac{\cos x_1 - x_1}{-\sin x_1 + 1}$

(D) $x_2 = x_1 + \dfrac{\cos x_1 - x_1}{\sin x_1 + 1}$

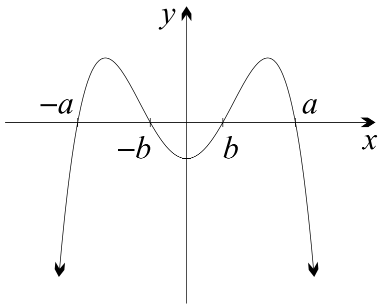
QUESTION SIX

Which diagram best represents $P(x) = (x-a)^2(b^2 - x^2)$, where $a > b$? **1**

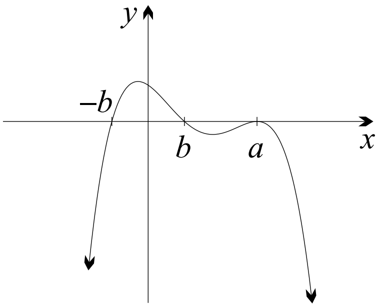
(A)



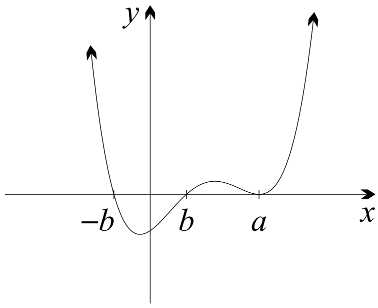
(B)



(C)



(D)



QUESTION SEVEN

What is the inverse function of $f(x) = \dfrac{5 + e^{2x}}{3}$? **1**

(A) $\dfrac{3}{5 + e^{2x}}$

(B) e^{5-3x}

(C) $\dfrac{1}{2}\ln(3x - 5)$

(D) $\dfrac{1}{2}\ln(5x - 3)$

Exam continues next page ...

QUESTION EIGHT

Consider the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers. One of the zeros of this polynomial is 1. What is the value of $a + b + c + d$? **1**

(A) -1

(B) $-1 - \alpha^2$

(C) $-2 - \alpha^2$

(D) 0

QUESTION NINE

An expression for the general solution to the trigonometric equation $\tan 3x = -\sqrt{3}$ is: **1**

(A) $x = \dfrac{n\pi}{3} - \dfrac{2\pi}{9}$ where n is any integer

(B) $x = \dfrac{n\pi}{3} + \dfrac{\pi}{3}$ where n is any integer

(C) $x = \dfrac{n\pi}{3} - \dfrac{\pi}{3}$ where n is any integer

(D) $x = \dfrac{n\pi}{3} + \dfrac{2\pi}{9}$ where n is any integer

QUESTION TEN

A ball is thrown into the air from a point O, where $x = 0$, with an initial velocity of 25 m/s at an angle $\theta = \tan^{-1} \frac{3}{4}$ to the horizontal. If air resistance is neglected and the acceleration due to gravity is taken as -10 m/s^2, then the ball reaches its greatest height after: **1**

(A) 1·5 seconds

(B) 15 seconds

(C) $\dfrac{2}{3}$ of a second

(D) 3 seconds

End of Section I

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Factorise $a^3 + 27b^3$. **1**

(b) Differentiate $\sin^{-1} 5x$. **1**

(c) Find $\displaystyle\int \frac{dx}{36 + x^2}$. **1**

(d) Evaluate $\displaystyle\int_0^{\pi} \sin^2 x \, dx$. **2**

(e) Find $\displaystyle\int x\sqrt{2 + x^2}\, dx$ using the substitution $u = 2 + x^2$. **2**

(f) Solve $\dfrac{4}{x+1} < 3$. **2**

(g) The variable point $(3t, 4t^2)$ lies on a parabola. Find the Cartesian equation of this parabola. **2**

(h) Find the coefficient of x^4 in the expansion of $(3 + x^2)^5$. **2**

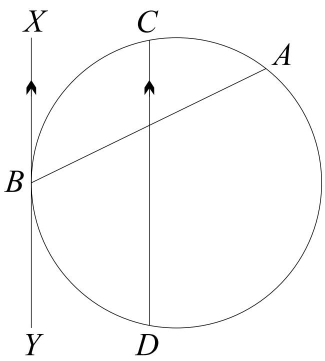
(i) Prove that $\tan(\frac{\pi}{4} + x) = \dfrac{\cos x + \sin x}{\cos x - \sin x}$, where $\cos x - \sin x \neq 0$. **2**

Exam continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

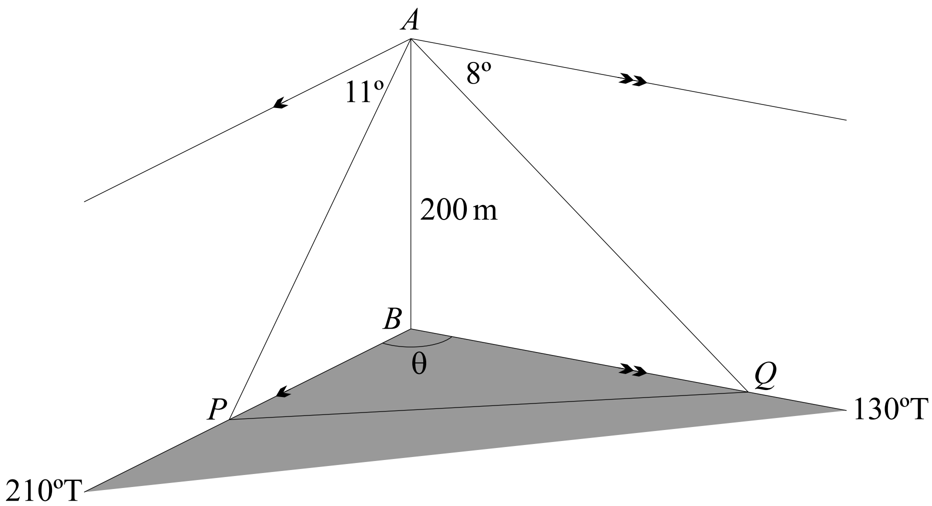
(a) Evaluate $\lim_{x \to 0} \dfrac{2\sin 3x}{5x}$. **1**

(b)



In the diagram above, AB and CD are intersecting chords. The tangent at B is parallel to CD. **3**
Copy this diagram into your exam booklet. Let $\angle XBC = \alpha$.
Prove that AB bisects $\angle CAD$.

(c)



HMAS Tarakan is enroute to carry out its latest mission in Cairns. It is first observed from the top of a 200 m cliff, AB at an angle of depression of 8°when it is at the point Q. Ten minutes later it is observed at point P with an angle of depression of 11°. Let $\angle PBQ = \theta$. The bearing of Q from B is 130° T and the bearing of P from B is 210° T.

(i) Show that $PQ^2 = 200^2(\tan^2 79^\circ + \tan^2 82^\circ - 2\tan 79^\circ \tan 82^\circ \cos\theta)$. **2**

(ii) Find the speed of HMAS Tarakan in km/h correct to three significant figures. **3**

Exam continues overleaf ...

QUESTION TWELVE (Continued)

(d) (i) Show that $\dfrac{d}{dx}\left(e^{4x}(\cos x - 4\sin x)\right) = -17e^{4x}\sin x$. **1**

(ii) Hence find $\displaystyle\int e^{4x}\sin x\,dx$. **1**

(e) The rate at which a body warms in air is proportional to the difference between its temperature T and the constant temperature A of the surrounding air. This rate can be expressed by the differential equation $\dfrac{dT}{dt} = k(T - A)$ where t is the time in minutes and k is a constant.

(i) Show that $T = A + Be^{kt}$, where B is a constant, is a solution of the differential equation. **1**

(ii) An object warms from 5°C to 15°C in 20 minutes. The temperature of the surrounding air is 25°C. Find the temperature of the object after a further 50 minutes have elapsed. Give your answer to the nearest degree. **3**

Exam continues next page ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) The polynomial $P(x) = ax^3 - 3x - 1$ has a remainder of -27 when divided by $(x+2)$.

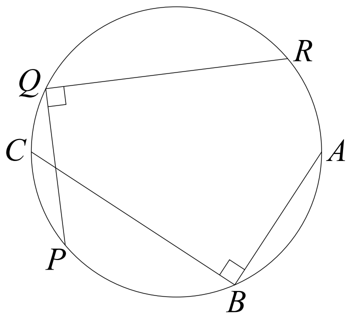
(i) Show that $a = 4$. **1**

(ii) Show that $(x - 1)$ is a factor of $P(x)$. **1**

(iii) Hence factorise $P(x)$ fully and sketch the curve $y = P(x)$ showing clearly all intercepts with the axes. **3**

(b) Use the principle of mathematical induction to show that $n^3 + 2n$ is divisible by 3 for all positive integers n. **3**

(c)



A, B, C, P, Q and R are points on a circle such that $\angle ABC$ and $\angle PQR$ are right angles.

Copy this diagram into your answer booklet.

(i) Explain why $PR = CA$. **1**

(ii) Prove that AP is equal and parallel to CR. **2**

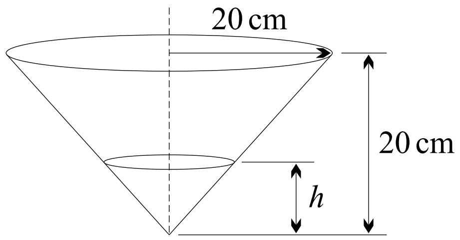
(d) Consider the identity of $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

(i) Show that $1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n = 0$. **1**

(ii) Show that $1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{1}{n}\binom{n}{n-1} = \frac{2(2^n - 1)}{n+1}$. **3**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



A conical vessel has height 20 cm and radius 20 cm. Water is poured into this vessel at a constant rate of 24 cm^3 per second. The depth of water is h cm at time t seconds.

(i) Show that the volume can be written $V = \frac{1}{3}\pi h^3$. **1**

(ii) What is the rate of increase of the cross-sectional area A of the surface of the liquid when the depth is 16 cm? **2**

(b) A particle moves along the x-axis starting at $x = 0{\cdot}5$. Its velocity, v metres per second, is described by $v = \sqrt{6x}\,e^{-x^2}$, where x is the displacement of the particle from the origin.

(i) Find the acceleration of the particle as a function of x. **2**

(ii) Find the maximum speed attained by the particle. **2**

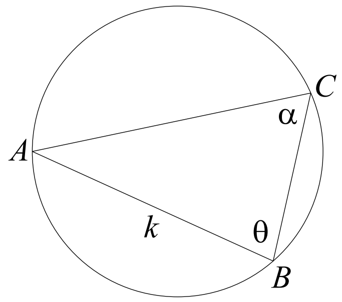
(iii) Show that T, the time taken to travel from $x = 1$ to $x = 3$, can be expressed as **1**

$$T = \frac{1}{\sqrt{6}} \int_1^3 x^{-\frac{1}{2}} e^{x^2}\, dx.$$

Do NOT evaluate this integral.

Exam continues next page ...

(c)



Points A, B and C lie on a circle as shown above. The length of the chord AB is a constant k. The sum of the lengths of the chords CA and CB is ℓ. Suppose that $\angle ABC = \theta$ radians and $\angle BCA = \alpha$ radians.

(i) Show that $\ell = \dfrac{k}{\sin\alpha}\big(\sin\theta + \sin(\theta+\alpha)\big)$. **3**

(ii) Explain why α is a constant. **1**

(iii) Show that $\dfrac{d\ell}{d\theta} = 0$ when $\theta = \dfrac{\pi}{2} - \dfrac{\alpha}{2}$. **2**

(iv) Hence show that the maximum value of ℓ occurs when $\theta = \dfrac{\pi}{2} - \dfrac{\alpha}{2}$. **1**

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} \, x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \;\; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} \, e^{ax}, \;\; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \;\; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

$$\text{NOTE: } \ln x = \log_e x, \; x > 0$$

Tear-off pages follow ...

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Thursday 9th August 2012

- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

CANDIDATE NUMBER: ....

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

Trial

1 D
2 A
3 A
4 B
5 D
6 C
7 C
8 A
9 D
10 A

Q11.

(a) $a^3+27b^3 = (a+3b)(a^2-3ab+9b^2)$. ✓

(b) $y = \sin^{-1} 5x$

$\frac{dy}{dx} = \frac{5}{\sqrt{1-25x^2}}$ or $\frac{1}{\sqrt{\frac{1}{25}-x^2}}$ ✓

(c) $\int \frac{dx}{36+x^2} = \frac{1}{6}\tan^{-1}\frac{x}{6} + c$ ✓

(d) $\int_0^{\pi} \sin^2 x \, dx = \frac{1}{2}\int_0^{\pi} (1-\cos 2x)\, dx$ ✓

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$

$= \frac{\pi}{2}$ ✓

(e) $\int x\sqrt{2+x^2}\, dx$

$u = 2+x^2$
$du = 2x\, dx$

$= \frac{1}{2}\int u^{\frac{1}{2}}\, du$ ✓

$= \frac{1}{2}\frac{2}{3}u^{3/2} + c$

$= \frac{1}{3}(2+x^2)^{3/2} + c$ ✓

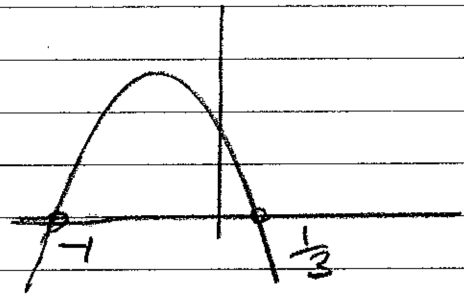
(does not matter if ext 2 students do not use subs.)

(f) $\frac{4}{x+1} < 3$

$4(x+1) < 3(x+1)^2$ ✓

$(x+1)[4 - 3(x+1)] < 0$

$(x+1)(1-3x) < 0$



$x < -1$ or $x > \frac{1}{3}$. ✓

(g) $x = 3t$ and $y = 4t^2$

so $t = \frac{x}{3}$ so $y = 4\left(\frac{x}{3}\right)^2$ ✓

$y = \frac{4x^2}{9}$ or $9y = 4x^2$ ✓

(h) ${}^5C_2\, 3^3 (x^2)^2$ is term in x^4.

So coeff is ${}^5C_2\, 3^3$

$= 270$ ✓✓

(✓ for ${}^5C_2$ and ✓ 3^3)

(i) $LHS = \tan\left(\frac{\pi}{4} + x\right)$

$= \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x}$

$= \frac{1 + \tan x}{1 - \tan x}$ ✓

$$= \frac{1 + \frac{\sin x}{\cos x}}{1 - \frac{\sin x}{\cos x}} \times \frac{\cos x}{\cos x}$$ ✓

$$= \frac{\cos x + \sin x}{\cos x - \sin x}$$

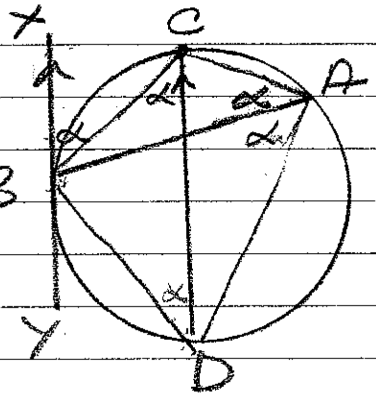
$= RHS$.

Q12.

(a) $\lim_{x \to 0} \frac{2\sin 3x}{5x} = 2 \times \frac{3}{5} \lim_{x \to 0} \frac{\sin 3x}{3x}$

$= \frac{6}{5}$ ✓

(b)



The angle between tangent and chord equals the angle in the alternate segment ✓

So $\angle XBA = \angle CAB = \alpha$

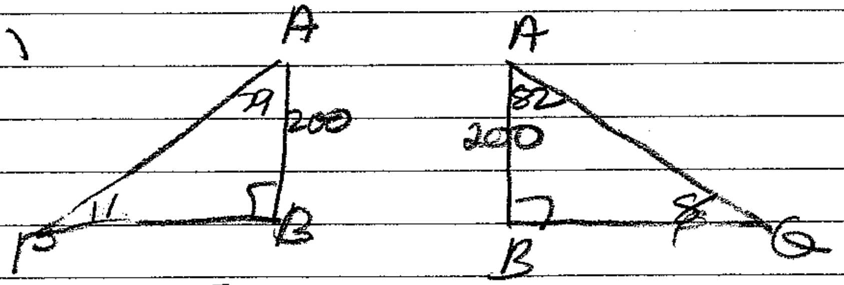
$\angle XBC = \angle BCD = \alpha$, alternate angles, $XY \parallel CD$ ✓

$\angle BCD = \angle BAD = \alpha$, angles in the same segment ✓

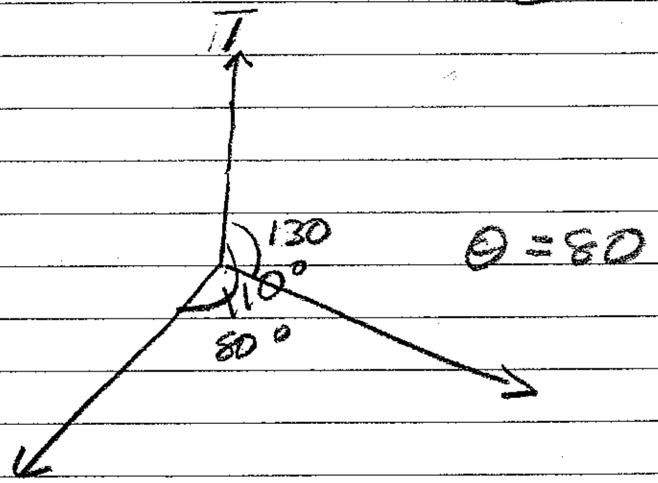
So $\angle BCD = \angle BAD = \alpha$

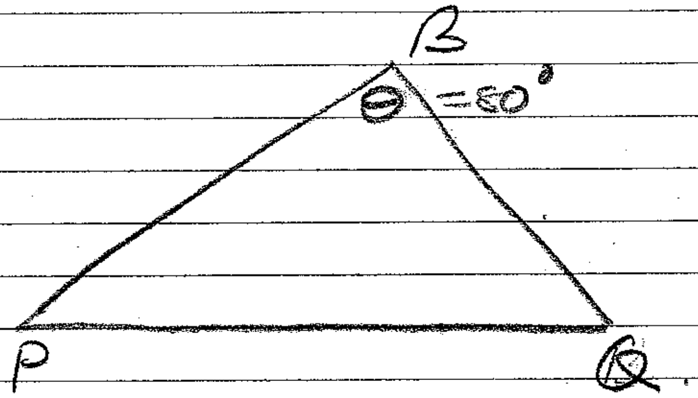
So AB bisects $\angle CAD$.

(c)



2 marks for showing where 79° & 82° come from — either diagram or writing.





(i) $\tan 79 = \frac{PB}{200}$ $\quad\quad$ $\tan 82 = \frac{BQ}{\tan 82}$

$PB = 200 \tan 79$ and $BQ = 200 \cot \beta$. ✓

(ii) Using cosine rule in $\triangle PBQ$.

$PQ^2 = (200 \tan 79)^2 + (200 \tan 82)^2 -$

$2 \times 200 \tan 79 \times 200 \tan 82 \times \cos \theta$

$= 200^2 (\tan^2 79° + \tan^2 82° - 2 \tan 79 \tan 82 \cdot \cos 80°)$

$= 2575279.955$

$PQ = 1604.76$ m ✓ $\quad$ ✓ for 80°

(iii) $\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{PQ}{10}$ km per min

$= \frac{1604.76}{10} \cdot \frac{60}{1000}$ km per hr.

$= 9.6$ km ph. ✓

(d)

(i) $y = e^{4x}(\cos x - 4\sin x)$ $u = e^{4x}$ $v = \cos x - 4\sin x$

$u' = 4e^{4x}$ $v' = -\sin x - 4\cos x$

$$\frac{dy}{dx} = vu' + uv'$$

$$= 4e^{4x}(\cos x - 4\sin x) + e^{4x}(-\sin x - 4\cos x)$$ ✓

$$= 4e^{4x}\cos x - 16e^{4x}\sin x - e^{4x}\sin x - 4e^{4x}\cos x$$

$$= -17e^{4x}\sin x$$

(ii) $\int -17e^{4x}\sin x\,dx = e^{4x}(\cos x - 4\sin x) + c$

so $\int e^{4x}\sin x\,dx = -\frac{1}{17}e^{4x}(\cos x - 4\sin x) + c$

(e)(i) $T = A + Be^{kt}$

$\frac{dT}{dt} = kBe^{kt}$, $Be^{kt} = T - A$ ✓

$= k(T-A)$

(ii) $t = 0$, $T = 5$, $A = 25$

$5 = 25 + B$

$B = -20$

So $T = 25 - 20e^{kt}$ ✓

$t = 20$, $T = 15$ gives $15 = 25 - 20e^{20k}$

$-10 = -20e^{20k}$

$e^{20k} = \frac{1}{2}$

$k = \frac{1}{20}\ln\frac{1}{2}$ ✓

$t = 70$, $T = 25 - 20e^{\frac{1}{20}\ln\frac{1}{2}(70)}$

$\simeq 23°$

Q13.

(a) (i) $P(x) = ax^3 - 3x - 1$

$$P(-2) = -8a + 6 - 1 = -27$$
$$-8a = -32$$
$$a = 4.$$

(ii) $P(x) = 4x^3 - 3x - 1$

$P(1) = 4 - 3 - 1 = 0$

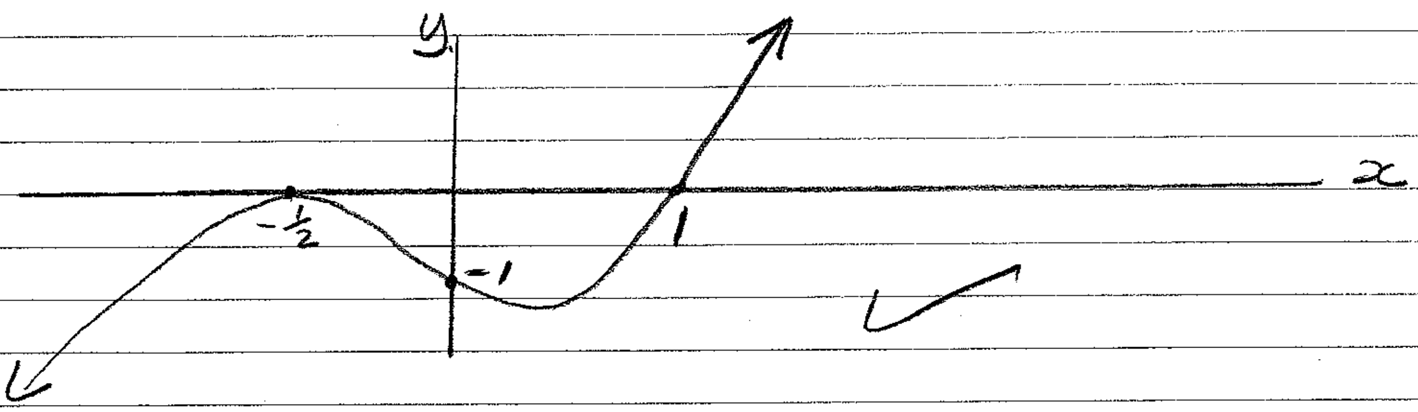
so $(x-1)$ is a factor

(iii)

$$\begin{array}{r} 4x^2 + 4x + 1 \\ x - 1 \,\big)\, 4x^3 - 0x^2 - 3x - 1 \\ \underline{4x^3 - 4x^2} \qquad\qquad \\ 4x^2 - 3x \qquad \\ \underline{4x^2 - 4x} \qquad \\ x - 1 \\ \underline{x - 1} \end{array}$$

now $4x^2 + 4x + 1 = (2x + 1)(2x + 1)$

so $4x^3 - 3x - 1 = (x-1)(2x+1)(2x+1)$.



(b)

A. Consider $n=1$, $1^3+2=3$ which is divisible by 3
So the statement is true for $n=1$.

B. Suppose that k^3+2k is divisible by 3 for some positive integer k

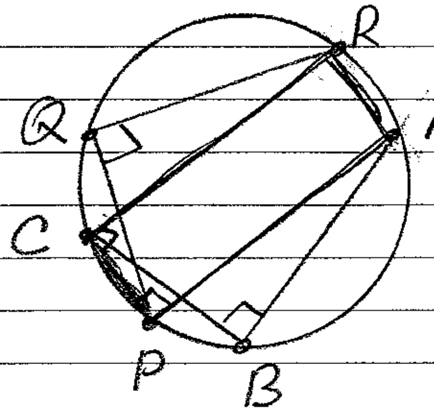
ie suppose that $k^3+2k=3M$, M an integer

Now prove that $(k+1)^3+2(k+1)$ is divisible by 3.

Now, $(k+1)^3+2(k+1) = k^3+3k^2+3k+1+2k+2$
$= k^3+2k + 3(k^2+k+1)$
$= 3M + 3(k^2+k+1)$ using the induction hypothesis
$= 3(M+k^2+k+1)$ which is clearly divisible by 3.

C. So by steps A & B and the principle of mathematical induction, the given statement is true.

c.



(i) PR and CA subtend right angles at the circumference, so they are both diagonals, and therefore equal. ✓

(ii) ∠RCP = 90° angle in semicircle, diameter PR.
∠CPA = 90° angle in semicircle, diameter CA
∠CRA = 180 − 90°, opposite ∠CBA in cyclic quad CBAR.
So CRAP is a rectangle (3 angles right angles)
So AP is equal and parallel to CR (opposite sides of rectangle).

✓✓ there are many ways to do this!!!

(d)

(i) $(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n$

let $x = -1$

Then $0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \binom{n}{n}(-1)^n$

But $\binom{n}{0} = \binom{n}{n} = 1$. ✓

so $0 = 1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots (-1)^n$.

(ii) integrate both sides of the expansion.

$$\frac{(1+x)^{n+1}}{n+1} = \binom{n}{0}x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \frac{1}{4}\binom{n}{3}x^4 + \cdots$$

$$+ \frac{1}{n+1}\binom{n}{n}x^{n+1} + k.$$ ✓

Find k: let $x = 0$

Then $\frac{1}{n+1} = 0 + 0 + 0 + \cdots + k$.

so $k = \frac{1}{n+1}$. ✓

and $$\frac{(1+x)^{n+1}}{n+1} = \binom{n}{0}x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \frac{1}{4}\binom{n}{3}x^4 + \cdots$$

$$\cdots \frac{1}{n}\binom{n}{n-1}x^n + \frac{1}{n+1}\binom{n}{n}x^{n+1} + \frac{1}{n+1}$$

let $x = 1$.

$$\frac{2^{n+1}}{n+1} = 1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + \cdots \frac{1}{n}\binom{n}{n-1} + \frac{1}{n+1} + \frac{1}{n+1}$$

$$\frac{2^{n+1}}{n+1} - \frac{2}{n+1} = 1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + \cdots + \frac{1}{n}\binom{n}{n-1}$$ ✓

$$\frac{2(2^n - 1)}{n+1} = 1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + \cdots \frac{1}{n}\binom{n}{n-1}.$$

Q14.

(a) (i) Here $r = h$

so $V = \frac{1}{3}\pi r^2 h$
$= \frac{1}{3}\pi h^3$ ✓

(ii) $\frac{dA}{dt} = \frac{dA}{dV}\frac{dV}{dt}$ and $\frac{dA}{dV} = \frac{dA}{dh}\frac{dh}{dV}$.

$= \frac{dA}{dh}\frac{dh}{dV}\frac{dV}{dt}$

Now $V = \frac{1}{3}\pi h^3$ so $\frac{dV}{dr} = \pi h^2$

$A = \pi h^2$ so $\frac{dA}{dr} = 2\pi h$

✓ for using the chain rule properly.

So $\frac{dA}{dt} = 2\pi h \times \frac{1}{\pi h^2} \times 24$.

$r = 16$, $\frac{dA}{dt} = \frac{2}{16} \times 24$
$= 3$ cm² per sec. ✓

b)

(i) $v = \sqrt{6x}\, e^{-x^2}$

$v^2 = 6x\, e^{-2x^2}$ ✓

$\frac{1}{2}v^2 = 3x\, e^{-2x^2}$

$\ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx} = 3x.(-4x)\, e^{-2x^2} + 3e^{-2x^2}$

$= -12x^2 e^{-2x^2} + 3e^{-2x^2}$ ✓

$= 3e^{-2x^2}(1 - 4x^2).$

(ii) Fastest speed is when $\ddot{x} = 0$.

$1 - 4x^2 = 0$

$x = \frac{1}{2}$ or $-\frac{1}{2}$ but $x > 0.5$

so $x = \frac{1}{2}$. ✓

$x = \frac{1}{2}$, $v = \sqrt{3}\, e^{-\frac{1}{4}}$ ms^{-1} ✓

(iii) $\frac{dx}{dt} = \sqrt{6x}\, e^{-x^2}$

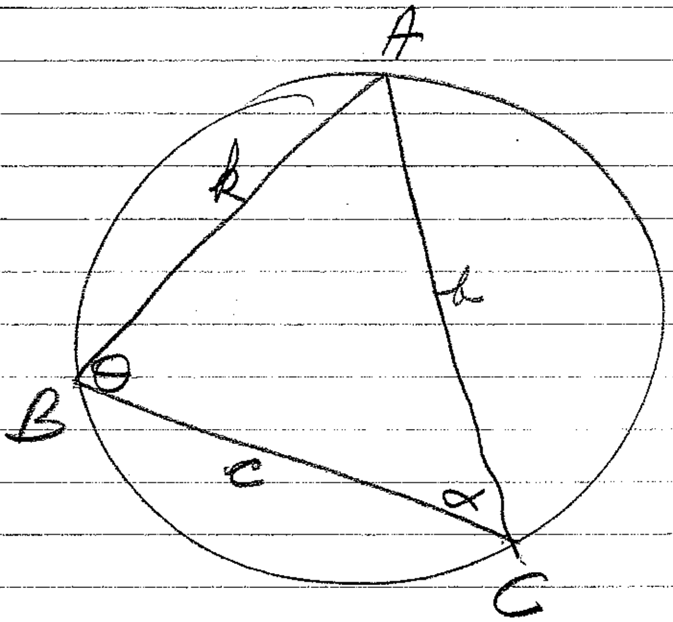
$\frac{dt}{dx} = \frac{1}{\sqrt{6x}}\, e^{x^2}$

$T = \int_1^3 \frac{1}{\sqrt{6x}}\, e^{x^2}\, dx$ ✓

$= \frac{1}{\sqrt{6}} \int_1^3 x^{-\frac{1}{2}} e^{x^2}\, dx.$

14.

(a)



(i) want $(c+b)$

Using sine rule in $\triangle ABC$

$$\frac{b}{\sin\theta} = \frac{k}{\sin\alpha} = \frac{c}{\sin(\pi-\theta-\alpha)}$$ ✓ use of sine rule.

so $b = \frac{k}{\sin\alpha}\sin\theta$

and $c = \frac{k}{\sin\alpha}\sin(\pi-\theta-\alpha)$

now $\ell = b + c$ ✓

$$= \frac{k}{\sin\alpha}\left(\sin\theta + \sin(\theta+\alpha)\right)$$

$$\sin(\pi-\theta-\alpha) = \sin(\pi-(\theta+\alpha))$$
$$= \sin\pi\cos(\theta+\alpha) - \sin(\theta+\alpha)\cos\pi$$
$$= 0 - -\sin(\theta+\alpha)$$
$$= \sin(\theta+\alpha)$$ ✓

(ii) α is a constant because chord AB is of constant length l, and chords of equal length subtend equal angles at the circumference. ✓

(iii) $$l = \frac{k}{\sin\alpha}\left(\sin\theta + \sin(\theta+\alpha)\right).$$

$$\frac{dl}{d\theta} = \frac{k}{\sin\alpha}\left(\cos\theta + \cos(\theta+\alpha)\right) ✓$$

at $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$

$$\frac{dl}{d\theta} = \frac{k}{\sin\alpha}\left[\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) + \cos\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)\right]$$

$$= \frac{k}{\sin\alpha}\left[\cos\frac{\pi}{2}\cos\frac{\alpha}{2} + \sin\frac{\pi}{2}\sin\frac{\alpha}{2} + \cos\frac{\pi}{2}\cos\frac{\alpha}{2} - \sin\frac{\pi}{2}\sin\frac{\alpha}{2}\right] ✓$$

$$= \frac{k}{\sin\alpha}\left[\,0\,\right]$$

$$= 0$$

(iv) $$\frac{d^2l}{d\theta^2} = \frac{k}{\sin\alpha}\left(-\sin\theta - \sin(\theta+\alpha)\right) = -l$$

since $l > 0$, this means that $\frac{d^2l}{d\theta^2} < 0$

so l is a max when $\frac{dl}{d\theta} = 0$ ie $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$ ✓